9.2 MONOTONE SEQUENCES

06/27

DEFINITION 9.2.1

A SEQUENCE $\{a_n\}_{n=1}^{+\infty}$ IS CALLED <u>STRICTLY INCREASING</u> IF:

$$a_1 < a_2 < a_3 < \dots < a_n < \dots$$

THE SEQUENCE IS <u>INCREASING</u> IF:

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq \dots$$

" " <u>STRICTLY DECREASING</u> IF:

$$a_1 > a_2 > a_3 > \dots > a_n > \dots$$

" " <u>DECREASING</u> IF:

$$a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$$

HERE IS AN EXAMPLE OF A SEQUENCE THAT IS STRICTLY DECREASING:

$$\left\{\frac{1}{n}\right\}_{n=1}^{+\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

<u>NOTE</u>: THIS SEQUENCE HAS A LOWER BOUND OF 0.

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{1}{n} = 0$$

THIS SEQUENCE CONVERGES TO 0.

AN INCREASING SEQUENCE LOOKS LIKE THE FOLLOWING:

$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$

A SEQUENCE THAT IS EITHER INCREASING OR DECREASING IS SAID TO BE MONOTONE.

IF A SEQUENCE IS STRICTLY INCREASING OR STRICTLY DECREASING, IT IS SAID TO BE STRICTLY MONOTONE.

TESTS FOR MONOTONICITY (TABLE 9.2.2):

	DIFFERENCE	RATIO
STRICTLY INCREASING	$a_{n+1} - a_n > 0$	$\frac{a_{n+1}}{a_n} > 1$
STRICTLY DECREASING	$a_{n+1} - a_n < 0$	$\frac{a_{n+1}}{a_n} < 1$
INCREASING	$a_{n+1} - a_n \geq 0$	$\frac{a_{n+1}}{a_n} \geq 1$
DECREASING	$a_{n+1} - a_n \leq 0$	$\frac{a_{n+1}}{a_n} \leq 1$

TABLE 9.2.3 DERIVATIVE OF $f(x)$ FOR $x \geq 1$

	CONCLUSION
$f'(x) > 0$	STRICTLY INCREASING
$f'(x) < 0$	STRICTLY DECREASING
$f'(x) \geq 0$	INCREASING
$f'(x) \leq 0$	DECREASING

DEFINITION 9.2.2

IF DISCARDING A FINITE NUMBER OF TERMS FROM THE BEGINNING OF A SEQUENCE PRODUCES A SEQUENCE WITH A CERTAIN PROPERTY, THEN THE ORIGINAL SEQUENCE IS SAID TO HAVE THAT PROPERTY **EVENTUALLY.**

$2, 1, \frac{1}{2}, \frac{1}{3}; 1, 2, 3, 4$

→ **EVENTUALLY STRICTLY INCREASING**

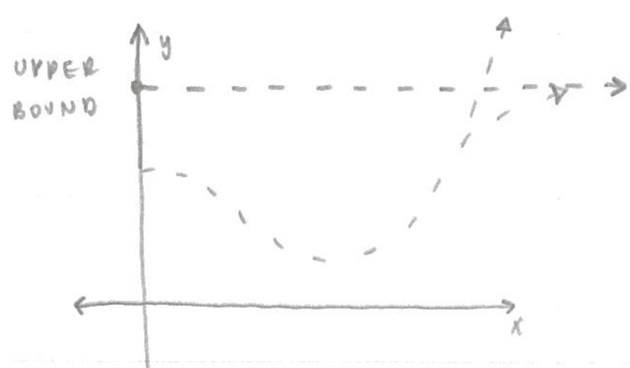
THEOREM 9.2.3

IF A SEQUENCE $\{a_n\}_{n=1}^{+\infty}$ IS EVENTUALLY INCREASING, THEN THERE ARE TWO POSSIBILITIES.

1) THERE IS AN UPPER BOUND "M" SUCH THAT $a_n \leq M$ FOR ALL n, IN WHICH CASE THE SEQUENCE CONVERGES TO $L \leq M$.

2) NO UPPER BOUND EXISTS.

$\lim_{n \to +\infty} a_n = +\infty$; DIVERGENT



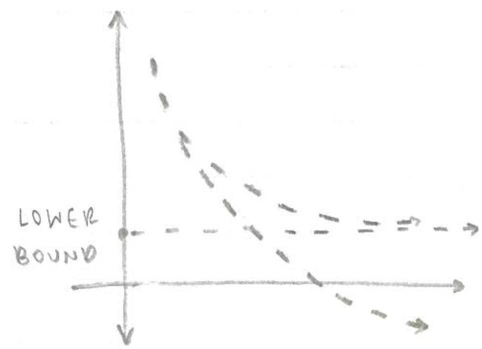
THEOREM 9.2.4

IF A SEQUENCE $\{a_n\}_{n=1}^{+\infty}$ IS EVENTUALLY DECREASING, THEN THERE ARE TWO POSSIBILITIES:

1) THERE IS A <u>LOWER BOUND</u> M SUCH THAT $a_n \geq M$ FOR ALL n, IN WHICH CASE THE SEQUENCE CONVERGES TO $L \geq M$.

2) NO LOWER BOUND EXISTS.

$\lim_{n \to +\infty} a_n = -\infty$; DIVERGENT



EX1) USE THE DIFFERENCE $a_{n+1} - a_n$ TO SHOW THE SEQUENCE IS STRICTLY INCREASING OR STRICTLY DECREASING.

$$\left\{ \frac{n}{2n+1} \right\}_{n=1}^{+\infty}$$

$$a_n = \frac{n}{2n+1}$$

$$a_{n+1} = \frac{n+1}{2(n+1)+1} = \frac{n+1}{2n+3}$$

$$\frac{n+1}{2n+3} - \frac{n}{2n+1}$$

$$= \frac{n+1\,(2n+1) - n\,(2n+3)}{(2n+3)(2n+1)}$$

$$= \frac{\cancel{2n^2} + \cancel{3n} + 1 - \cancel{2n^2} - \cancel{3n}}{(2n+3)(2n+1)}$$

$$= \frac{1}{(2n+3)(2n+1)} > 0, \quad n \geq 1$$

$$\boxed{\therefore \left\{ \frac{n}{2n+1} \right\}_{n=1}^{+\infty} \text{ IS STRICTLY INCREASING}}$$

EX2) USE $\frac{a_{n+1}}{a_n}$ TO SHOW $\left\{ \frac{n^n}{n!} \right\}_{n=1}^{+\infty}$ IS STRICTLY MONOTONE.

$$a_n = \frac{n^n}{n!}$$

$$a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \frac{(n+1)^{n+1}}{n+1!\,\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{n^n}$$

$$= \frac{(n+1)^{n\cancel{+1}}}{\cancel{n+1}} \cdot \frac{1}{n^n}$$

$$= \frac{(n+1)^n}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n$$

$$= \left(1 + \frac{1}{n}\right)^n > 1 \text{ FOR ALL } n \geq 1$$

$\therefore \left\{ \frac{n^n}{n!} \right\}_{n=1}^{+\infty}$ IS STRICTLY INCREASING

EX3) DETERMINE THE MONOTONICITY OF THE SEQUENCE $\left\{ \frac{2^n}{1+2^n} \right\}_{n=1}^{+\infty}$

HERE, WE WILL USE $\frac{a_{n+1}}{a_n}$

$$a_n = \frac{2^n}{1+2^n}$$

$$a_{n+1} = \frac{2^{n+1}}{1+2^{n+1}}$$

$$= \frac{\cancel{2^{n+1}}\,2}{1+2^{n+1}} \cdot \frac{1+2^n}{\cancel{2^n}\,1}$$

$$= \frac{2(1+2^n)}{(1+2^{n+1})}$$

$$= \frac{(2+2^{n+1})}{(1+2^{n+1})}$$

$$= \frac{1+1+2^{n+1}}{1+2^{n+1}}$$

$$= \frac{1}{1+2^{n+1}} + 1$$

$$1 + \frac{1}{1+2^{n+1}} > 1$$

FOR ALL $n \geq 1$

$$\boxed{\therefore \left\{ \frac{2^n}{1+2^n} \right\}_{n=1}^{+\infty} \text{ IS STRICTLY INCREASING}}$$

EX4) DETERMINE THE MONOTONICITY OF

$$\left\{ ne^{-2n} \right\}_{n=1}^{+\infty}$$

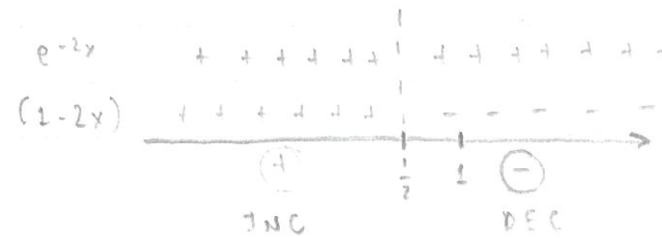
LET $f(x) = xe^{-2x}$ $\quad x \geq 1$

$$f'(x) = 1 \cdot e^{-2x} + x \cdot e^{-2x} \cdot -2$$

$$= e^{-2x} - 2xe^{-2x}$$

$$= \underline{\underline{e^{-2x}(1-2x)}}$$

USING A SIGN CHART TO ANALYZE $f'(x)$



$f'(x) < 0$ $\quad x \geq 1$

$$\boxed{\therefore \left\{ ne^{-2n} \right\}_{n=1}^{+\infty} \text{ IS STRICTLY DECREASING}}$$