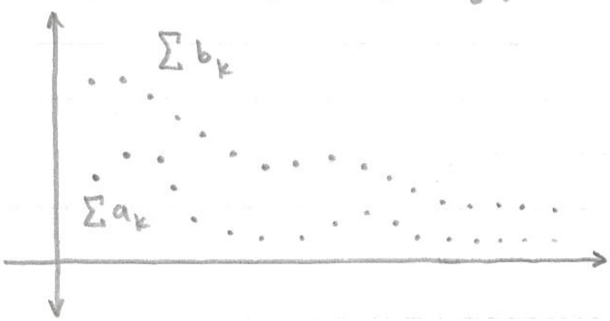
9.5 THE COMPARISON, RATIO, AND ROOT TESTS

06/30

THEOREM 9.5.1 THE COMPARISON TEST

GIVEN $\sum a_k$ AND $\sum b_k$ WITH NON-NEGATIVE TERMS WHERE

$a_1 \leq b_1,\ a_2 \leq b_2,\ a_3 \leq b_3, \ldots$



IF $\sum b_k$ CONVERGES, THEN $\sum a_k$ CONVERGES.

IF $\sum a_k$ DIVERGES, THEN $\sum b_k$ DIVERGES.

9.5.2 INFORMAL PRINCIPLE

CONSTANT TERMS IN THE DENOMINATOR OF u_k CAN USUALLY BE DELETED WITHOUT AFFECTING CONVERGENCE OR DIVERGENCE.

$$\sum_{k=1}^{+\infty} \frac{1}{k^2 - 5} \approx \sum_{k=1}^{+\infty} \frac{1}{k^2}$$

9.5.3 INFORMAL PRINCIPLE

IF A POLYNOMIAL IN k APPEARS AS A FACTOR IN THE NUMERATOR OR DENOMINATOR OF u_k, ALL BUT THE LEADING TERM IN THE POLYNOMIAL CAN USUALLY BE DELETED WITHOUT AFFECTING CONVERGENCE OR DIVERGENCE.

$$\sum_{k=1}^{+\infty} \frac{k^2 - 5k + 5}{k^4 - 3k^3 + 2k - 5} \approx \sum_{k=1}^{+\infty} \frac{1}{k^2}$$

EX 1) DETERMINE IF THE SERIES CONVERGES OR DIVERGES.

$$\sum_{k=4}^{+\infty} \frac{1}{k-3}$$

NOTE: WHICH IS GREATER?

$$\frac{7}{20} \quad \frac{9}{23}$$

$$\frac{161}{460} < \frac{180}{460}$$

$$\frac{1}{k-3} > \frac{1}{k}$$

SINCE $\sum \frac{1}{k}$ IS A DIVERGENT HARMONIC SERIES,

$\sum_{k=4}^{+\infty} \frac{1}{k-3}$ IS DIVERGENT BY THE COMPARISON TEST.

EX2) USE THE COMPARISON TEST TO SHOW IF THE SERIES CONVERGES OR DIVERGES.

a) $\sum_{k=1}^{+\infty} \frac{1}{\sqrt{k-1}}$

CROSS PRODUCTS

$$\sqrt{k} > \sqrt{k-1}$$

$\sum_{k=1}^{+\infty} \frac{1}{\sqrt{k}} \Rightarrow$ p-SERIES

$$\frac{1}{\sqrt{k-1}} > \frac{1}{\sqrt{k}} \qquad k \geq 1$$

$p = \frac{1}{2} < 1$

DIVERGENT

$$\boxed{\therefore \sum_{k=1}^{+\infty} \frac{1}{\sqrt{k-1}} \text{ IS DIVERGENT BY THE C.T.}}$$

b) $\sum_{k=1}^{+\infty} \frac{1}{k^2+5}$

$k^2 < k^2+5$ ✓

$\sum_{k=1}^{+\infty} \frac{1}{k^2} \Rightarrow$ p-SERIES

$$\frac{1}{k^2+5} < \frac{1}{k^2}$$

$p = 2 > 1$

CONVERGENT

$$\boxed{\therefore \sum_{k=1}^{+\infty} \frac{1}{k^2+5} \text{ IS CONVERGENT BY THE C.T.}}$$

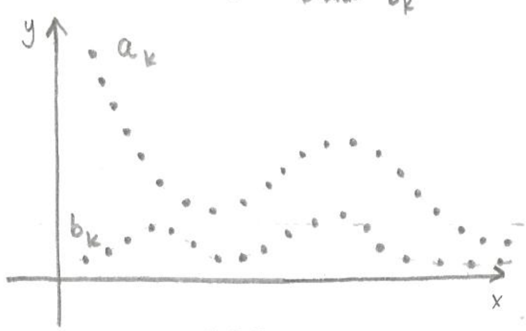
THEOREM 9.5.4 THE LIMIT COMPARISON TEST (L.C.T.)

LET Σa_k AND Σb_k BE SERIES WITH POSITIVE TERMS AND

$$\boxed{\rho = \lim_{k \to +\infty} \frac{a_k}{b_k}}$$

IF ρ IS FINITE AND $\rho > 0$, THEN BOTH SERIES CONVERGE OR BOTH DIVERGE.

NOTE: SUPPOSE $\rho = \lim_{k \to +\infty} \frac{a_k}{b_k} = 2$



EX3) DETERMINE IF THE SERIES CONVERGES OR DIVERGES.

a) $\displaystyle\sum_{k=1}^{+\infty} \left(\frac{1}{k+5}\right)$

USING THE L.C.T. WITH

$\displaystyle\sum_{k=1}^{+\infty} \frac{1}{k}$; DIVERGENT HARMONIC SERIES

$$\rho = \lim_{k \to +\infty} \frac{a_k}{b_k}$$

$$= \lim_{k \to +\infty} \frac{\frac{1}{k+5}}{\frac{1}{k}}$$

$$= \lim_{k \to +\infty} \frac{k}{k+5} \cdot \frac{\frac{1}{k}}{\frac{1}{k}}$$

$$= \lim_{k \to +\infty} \frac{1}{1 + \cancel{\frac{5}{k}}^{\,0}}$$

$$= 1 \quad \text{FINITE}$$

$\therefore \displaystyle\sum_{k=1}^{+\infty} \frac{1}{k+5}$ IS DIVERGENT BY THE L.C.T.

b) $\sum_{k=1}^{+\infty} \frac{1}{2k^2+3k+4}$

USING THE L.C.T. WITH

$\sum_{k=1}^{+\infty} \frac{1}{k^2} \Rightarrow$ P-SERIES

$p = 2 > 1 \rightarrow$ CONVERGENT

$$\rho = \lim_{k \to +\infty} \frac{a_k}{b_k}$$

$$= \lim_{k \to +\infty} \frac{\frac{1}{2k^2+3k+4}}{\frac{1}{k^2}}$$

$$= \lim_{k \to +\infty} \frac{k^2}{2k^2+3k+4} \cdot \frac{\frac{1}{k^2}}{\frac{1}{k^2}}$$

$= \frac{1}{2}$ FINITE

$\boxed{\therefore \sum_{k=1}^{+\infty} \frac{1}{2k^2+3k+4} \text{ IS CONVERGENT BY L.C.T.}}$

THEOREM 9.5.5 THE RATIO TEST

SUPPOSE $\sum u_k$ HAS POSITIVE TERMS AND

$$* \boxed{\rho = \lim_{k \to +\infty} \left[\frac{u_{k+1}}{u_k} \right]}$$

a) IF $\rho < 1$, $\sum u_k$ CONVERGES.

b) IF $\rho > 1$, $\sum u_k$ DIVERGES.

c) IF $\rho = 1$, INCONCLUSIVE.

EX4) DETERMINE IF THE SERIES CONVERGES OR DIVERGES.

$$\sum_{k=1}^{+\infty} \frac{k!}{k^k}$$

USING THE RATIO TEST.

$$u_k = \frac{k!}{k^k}$$

$$u_{k+1} = \frac{(k+1)!}{(k+1)^{k+1}}$$

$$\rho = \lim_{k \to +\infty} \left[\frac{u_{k+1}}{u_k} \right]$$

$$= \lim_{k \to +\infty} \left[\frac{(k+1)\cancel{!}}{(k+1)^{k+1}} \cdot \frac{k^k}{\cancel{k!}} \right]$$

$$= \lim_{k \to +\infty} \left[\frac{(k+1)}{(k+1)^{k+1}} \cdot k^k \right]$$

$$= \lim_{k \to +\infty} \frac{k^k}{(k+1)^k}$$

$$= \lim_{k \to +\infty} \left[\frac{k}{k+1} \right]^k$$

$$= \lim_{k \to +\infty} \left[\frac{k+1}{k} \right]^{-k}$$

$$= \lim_{k \to +\infty} \left[1 + \frac{1}{k} \right]^{-k}$$

$$= \lim_{k \to +\infty} \left[\frac{1}{\left\{ 1 + \frac{1}{k} \right\}^k} \right]$$

$$= \frac{1}{\lim_{k \to +\infty} \left(1 + \frac{1}{k} \right)^k}$$

$$= \frac{1}{e} < 1$$

$$\boxed{\therefore \sum_{k=1}^{+\infty} \frac{k!}{k^k} \text{ IS CONVERGENT BY THE RATIO TEST}}$$

THEOREM 9.5.6 THE ROOT TEST

LET $\sum u_k$ BE A SERIES WITH POSITIVE TERMS AND

$$* \boxed{\rho = \lim_{k \to +\infty} \sqrt[k]{u_k} = \lim_{k \to +\infty} (u_k)^{\frac{1}{k}}}$$

a) $\rho < 1$, $\sum u_k$ CONVERGES.

b) $\rho > 1$, $\sum u_k$ DIVERGES.

c) $\rho = 1$, INCONCLUSIVE.

EX5) DETERMINE IF THE SERIES CONVERGES OR DIVERGES.

a) $\displaystyle\sum_{k=1}^{+\infty} \frac{k}{5^k}$

USING THE ROOT TEST

$$\rho = \lim_{k \to +\infty} (u_k)^{\frac{1}{k}}$$

$$= \lim_{k \to +\infty} \left[\frac{k}{5^k}\right]^{\frac{1}{k}}$$

$$= \lim_{k \to +\infty} \left[\frac{k^{\frac{1}{k}}}{5}\right]$$

$$= \frac{1}{5} \underbrace{\lim_{k \to +\infty} \left[k^{\frac{1}{k}}\right]}$$

$$= \frac{1}{5} \cdot 1$$

$$= \frac{1}{5}$$

let $y = k^{\frac{1}{k}}$

$\ln y = \ln k^{\frac{1}{k}}$

$$\lim_{k \to +\infty} \ln y = \lim_{k \to +\infty} \left[\frac{\ln k}{k}\right]$$

LET $f(x) = \frac{\ln y}{x}$ $\quad x \geq 1$

$$= \lim_{x \to +\infty} \frac{\ln x}{x}$$

$$= \lim_{x \to +\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\therefore \lim_{k \to +\infty} y = e^0 = 1$$

$$\therefore \rho = \frac{1}{5} \cdot 1 = \frac{1}{5} < 1$$

$\therefore \displaystyle\sum_{k=1}^{+\infty} \frac{k}{5^k}$ IS CONVERGENT BY THE ROOT TEST

b) $\sum_{k=1}^{+\infty} \frac{5\sin^2 k}{k!}$

NOTE: $0 < \sin^2 k < 1$

$$\frac{5\sin^2 k}{k!} < \frac{5}{k!} \qquad k \geq 1$$

$$\sum_{k=1}^{+\infty} \frac{5}{k!}$$

USING THE RATIO TEST

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k}$$

$$= \lim_{k \to +\infty} \left[\frac{5}{(k+1)!} \cdot \frac{k!}{5} \right]$$

$$= \lim_{k \to +\infty} \left[\frac{1}{k+1} \right]$$

$$= 0 < 1$$

$\therefore \sum_{k=1}^{+\infty} \frac{5}{k!}$ IS CONVERGENT BY THE RATIO TEST

$\boxed{\therefore \sum_{k=1}^{+\infty} \frac{5\sin^2 k}{k!} \text{ IS CONVERGENT BY THE COMPARISON TEST}}$