* WOLFRAM ALPHA

* INTEGRAL CALCULATOR

7.3 INTEGRATING TRIGONOMETRIC FUNCTIONS

EX 1) a) $\int \sin^2\theta \, d\theta = \frac{1}{2}\int (1-\cos 2\theta)\, d\theta$

$= \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$

$= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C$

$= \frac{\theta}{2} - \frac{\sin\theta\cos\theta}{2} + C$

NOTE: $\boxed{\sin^2 x = \frac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)}$

RECALL: $\sin 2\theta = 2\sin\theta\cos\theta$

b) $\int \sin^3\theta \, d\theta = \int \sin^2\theta \sin\theta \, d\theta$ ← ISOLATE A $\sin\theta$ (ODD POWERS)

$= \int (1-\cos^2\theta)\sin\theta \, d\theta = -\int (1-u^2)\, du$

let $u = \cos\theta$

$du = -\sin\theta \, d\theta$

$-du = \sin\theta \, d\theta$

$= -\left(u - \frac{u^3}{3}\right)$

$= \boxed{-\cos\theta + \frac{\cos^3\theta}{3} + C}$

INTEGRALS OF THE FORM

$\int \sin^m\theta \cos^n\theta \, d\theta$

EX 2) $\int \sin^4\theta \cos^3\theta \, d\theta = \int \sin^4\theta \cos^2\theta \cos\theta \, d\theta$ ← ALWAYS WANT EVEN POWERS

$= \int \sin^4\theta (1-\sin^2\theta)\cos\theta \, d\theta = \int u^4(1-u^2)\, du$

let $u = \sin\theta$

$du = \cos\theta \, d\theta$

$= \int u^4 - u^6 \, du$

$= \frac{u^5}{5} - \frac{u^7}{7} + C$

$= \boxed{\frac{\sin^5\theta}{5} - \frac{\sin^7\theta}{7} + C}$

EX 3) $\int \sin^5 x \cos^2 x \, dx = \int \sin^4 x \cos^2 x \cdot \sin x \, dx$

$= \int (1-\cos^2 x)^2 \cos^2 x \sin x \, dx = -\int (1-u^2)^2 u^2 \, du$

let $u = \cos x$

$-du = \sin x \, dx$

$= -\int (1-2u^2+u^4) u^2 \, du$

$= -\int u^6 - 2u^4 + u^2 \, du$

$= -\left[\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} \right] + C$

$= \boxed{-\frac{\cos^7 x}{7} + \frac{2\cos^5 x}{5} - \frac{\cos^3 x}{3} + C}$

WHAT DO WE DO WHEN m & n ARE BOTH ODD?

EX 4) $\int \sin^3 x \cos^3 x \, dx = \int \sin^3 x \cos^2 x \cos x \, dx$

$= \int \sin^3 x (1-\sin^2 x) \cos x \, dx$

let $u = \sin x$

$du = \cos x \, dx$

$= \int u^3 (1-u^2) \, du = \int u^3 - u^5 \, du = \frac{u^4}{4} - \frac{u^6}{6} + C$

$= \boxed{\frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C}$

EX5) $\int \sin^2\theta \cos^2\theta \, d\theta$ -OR- $\int (\sin x \cos x)^2 \, dx$

$= \int \frac{1}{2}(1-\cos 2\theta) \cdot \frac{1}{2}(1+\cos 2\theta)$

$= \frac{1}{4}\int (1-\cos^2 2\theta) \, d\theta$

$= \frac{1}{4}\int d\theta - \frac{1}{4}\int \frac{1}{2}(1+\cos 4\theta) \, d\theta$

$= \frac{\theta}{4} - \frac{1}{8}\int 1+\cos 4\theta \, d\theta$

$= \boxed{\frac{\theta}{4} - \frac{\theta}{8} - \frac{\sin 4\theta}{32} + C}$

$= \frac{2\theta}{8} - \frac{\theta}{8} - \frac{\sin 4\theta}{32} + C$

$= \boxed{\frac{\theta}{8} - \frac{\sin 4\theta}{32} + C}$ ✓

$= \int \left[\frac{\sin 2x}{2}\right]^2 dx$

$= \frac{1}{4}\int \sin^2 2x \, dx$

$= \frac{1}{4}\int \frac{1}{2}(1-\cos 4x) \, dx$

$= \frac{1}{8}\int 1-\cos 4x \, dx$

$= \boxed{\frac{\theta}{8} - \frac{\sin 4x}{32} + C}$ ✓

EX6) $\int \sec x \, dx = \int \frac{\sec x}{1} \cdot \left[\frac{\sec x + \tan x}{\sec x + \tan x}\right]$

$= \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} \, dx = \int \frac{1}{u} \, du = \ln|u| + C$

let $u = \tan x + \sec x$

$du = \sec^2 x + \tan x \sec x$

$\bigstar = \boxed{\ln|\sec x + \tan x| + C}$

SIMILARLY:

$\bigstar$ $\boxed{\int \csc x \, dx = \ln|\csc x - \cot x| + C}$

WHAT ABOUT $\int \sec^2 x \, dx$?

$= \boxed{\tan x + C}$

EX 7) $\int \sec^3 x \, dx$

$$= \int \sec^2 x \sec x \, dx = \sec x \tan x - \int \underbrace{\tan^2 x \sec x}_{(\sec^2 x - 1)} \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$+ \int \sec^3 dx$ $\qquad$ $+ \int \sec^3 x \, dx$

	~~u~~	~~dv~~
$(+)$	~~$\sec^2 x$~~	~~$\sec x$~~
$(-)$	~~$2\sec^2 x \tan x$~~	~~$\ln\lvert\sec x + \tan x\rvert$~~

	u	dv
$(+)$	$\sec x$	$\sec^2 x$
$(-)$	$\sec x \tan x$	$\tan x$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln\lvert\sec x + \tan x\rvert + C$$

$$= \boxed{\frac{\sec x \tan x + \ln\lvert\sec x + \tan x\rvert}{2} + C}$$

EX 8) $\int \sec^4 x \, dx$

$$= \int \sec^2 x \sec^2 x \, dx = \int (1 + \tan^2 x) \sec^2 x \, dx = \int 1 + u^2 \, du$$

let $u = \tan x$

$du = \sec^2 x \, dx$

$$= u + \frac{u^3}{3} + C$$

$$= \boxed{\tan x + \frac{\tan^3 x}{3} + C}$$

EX 9) $\int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du = -\ln|u| + C = \boxed{-\ln|\cos x| + C}$ ☆

let $u = \cos x$

$-du = \sin x \, dx$

– OR

$= \boxed{\ln|\sec x| + C}$

EX10) $\int \tan^2 x \, dx$

$\int (\sec^2 x - 1) dx = \boxed{\tan x - x + C}$

EX11) $\int \tan^3 x \, dx$

☆NOTE: ALWAYS TRY TO GET A $\sec^2 x$ W/ EACH $\tan x$ TERM TO MAKE A $\tan x$ "U"-SUB

$= \int \tan^2 x \tan x \, dx$

$= \int (\sec^2 x - 1) \tan x \, dx$

$= \int \sec^2 x \tan x - \tan x \, dx = \int \sec^2 x \tan x - \int \tan x \, dx$

let $u = \tan x$

$du = \sec^2 x$

$= \int u \, du - \int \frac{\sin x}{\cos x} dx$

$= \frac{u^2}{2} - [-\ln|\cos x|] + C$

$= \boxed{\frac{\tan^2 x}{2} + \ln|\cos x| + C}$

EX 12) $\int \tan^4 x \, dx$

$= \int \tan^2 x \tan^2 x \, dx$

$= \int (\sec^2 x - 1) \tan^2 x \, dx$

$= \int \sec^2 x \tan^2 x \, dx - \int \tan^2 x \, dx$

let $u = \tan x$

$du = \sec^2 x \, dx$

$= \int u^2 \, du - \int (\sec^2 x - 1) \, dx$

$= \frac{u^3}{3} - (\tan x - x) + C$

$= \boxed{\frac{\tan^3 x}{3} - \tan x + x + C}$

EX 13) $\int \tan^2 x \tan^3 x \, dx$

$= \int (\sec^2 x - 1) \tan^3 x \, dx$

$= \int \tan^3 x \sec^2 x - \int \underbrace{\tan^3 x}_{\tan^2 x \cdot \tan x} \, dx$

$= \int \tan^3 x \sec^2 x - \int (\sec^2 x - 1) \tan x \, dx$

$= \int \tan^3 x \sec^2 x - \int \sec^2 x \tan x + \int \tan x \, dx$ → **NOTE**: $u = \tan x$, $du = \sec^2 x$

$= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + [-\ln|\cos x|] + C$

$= \int u^3 \, du - \int u \, du$

$= \boxed{\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \ln|\cos x| + C}$

$= \frac{u^4}{4} - \frac{u^2}{2}$

HWK #19) $\int_0^{\frac{\pi}{3}} \sin^4 3x \cos^3 3x \, dx$

$= \int_0^{\frac{\pi}{3}} \sin^4 3x \cos^2 3x \cos 3x \, dx$

$= \int_0^{\frac{\pi}{3}} \sin^4 3x \cdot (1 - \sin^2 3x) \cos 3x \, dx$

let $u = \sin 3x$ $\qquad x = \frac{\pi}{3} \rightarrow u = \sin 3 \cdot \frac{\pi}{3} = 0$

$du = \cos 3x \cdot 3 \, dx$ $\qquad x = 0 \rightarrow u = \sin 3 \cdot 0 = 0$

$\frac{1}{3} du = \cos 3x \, dx$

$= \frac{1}{3} \int_0^0 u^4 (1 - u^2) \, du$

$= 0$

HWK #13) $\int \sin 2x \cos 3x \, dx$ $\qquad$ **NOTE**: $\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$

$= \frac{1}{2} \int (\sin(-x) + \sin 5x) \, dx$

$\boxed{\begin{aligned} \sin(-x) &= -\sin x \\ \cos(-x) &= \cos x \end{aligned}}$ ODD / EVEN

$= \frac{1}{2} \int -\sin x \, dx + \frac{1}{2} \int \sin 5x \, dx$

$= \frac{\cos x}{2} + \frac{1}{2} \left[- \frac{\cos 5x}{5} \right] + C$

$= \boxed{\frac{\cos x}{2} - \frac{\cos 5x}{10} + C}$