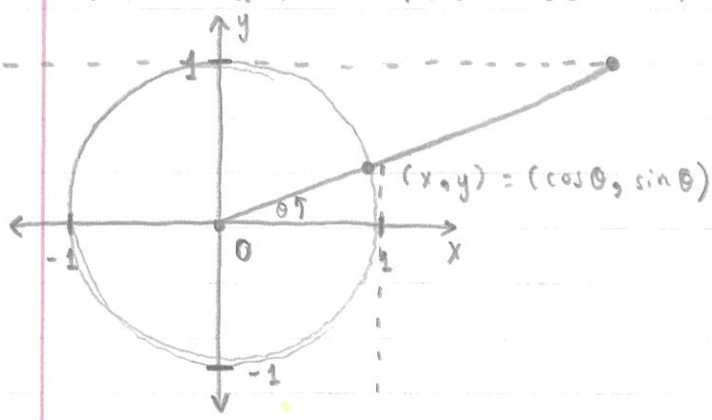
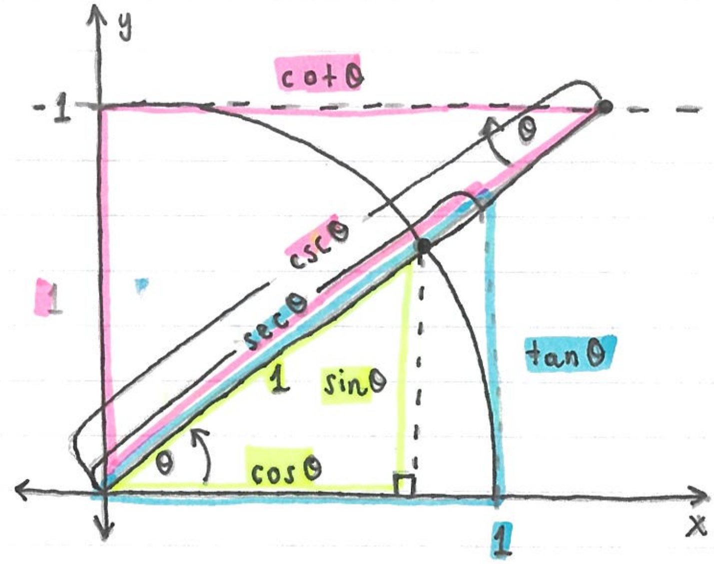
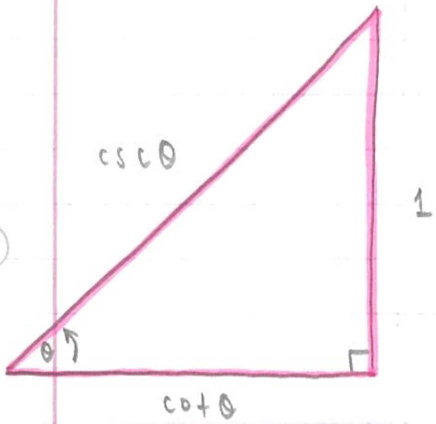
7.4 TRIGONOMETRIC SUBSTITUTION

6/14

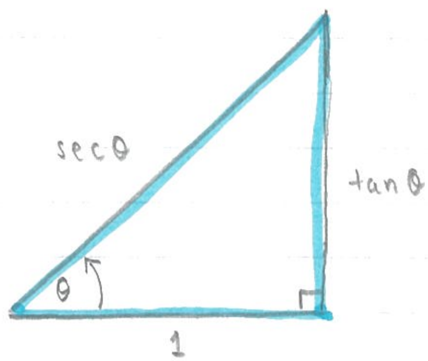


$x^2 + y^2 = 1$

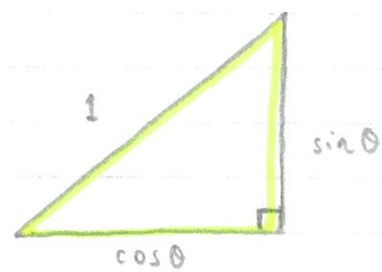




$\cot^2\theta + 1 = \csc^2\theta$



$\tan^2\theta + 1 = \sec^2\theta$



$\cos^2\theta + \sin^2\theta = 1$

NOTE THE FOLLOWING THREE CASES:

1) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \frac{\cos\theta}{1} \cdot d\theta = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta = \int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta$

$\displaystyle = \int \frac{\cos\theta}{\cos\theta}\,d\theta$

$\displaystyle = \int d\theta$

$= \theta + C$

WE WANT $x^2 = \sin^2\theta$, SO

WE LET $x = \sin\theta$ ← TRIG SUBSTITUTION

$dx = \cos\theta\, d\theta$

SINCE $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$\therefore\ \theta + C = \boxed{\sin^{-1}x + C}$

2) $\int \frac{1}{\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta-1}}\,d\theta = \int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\,d\theta$

$$= \int \frac{\sec\theta\,\cancel{\tan\theta}}{\cancel{\tan\theta}}\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

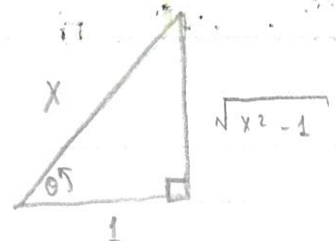
$$= \boxed{\ln\left|x + \sqrt{x^2-1}\right| + C}$$

HERE, WE WANT $x^2 = \sec^2\theta$

LET $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

REFERENCE TRIANGLE



* NOTE: USE REFERENCE TRIANGLE TO GET APPROPRIATE VALUES IN ANSWER

3) $\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{\tan^2\theta+1}} \cdot \frac{\sec^2\theta}{1}\,d\theta = \int \frac{\sec\theta}{\sqrt{\sec^2\theta}}\,d\theta$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

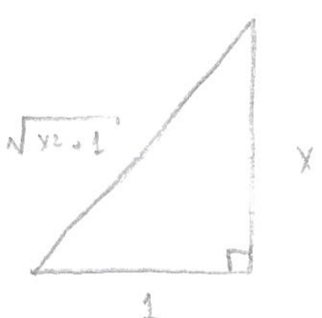
$$= \boxed{\ln\left|\sqrt{x^2+1} + x\right| + C}$$

WE WANT $x^2 = \tan^2\theta$;

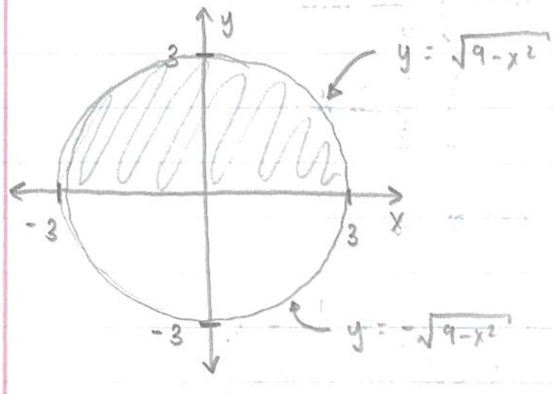
LET $x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

REFERENCE Δ



EX1) FIND THE AREA OF A CIRCLE OF RADIUS 3 USING INTEGRATION.



$$A = 2\int_{-3}^{3} \sqrt{9-x^2}\,dx$$

$$= 4\int_{0}^{3} \sqrt{9-x^2}\,dx$$

NOTE: DOUBLE AREA B/C WE HAVE HALVES

$x^2 + y^2 = 9$

$y^2 = 9 - x^2$

$y = \pm\sqrt{9-x^2}$

WANT $x^2 = 9\sin^2\theta$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

NOTE: MUST ALSO CHANGE LIMITS OF INTEGRATION

$$= 4\int_{0}^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,d\theta$$

$$= 4\int_{0}^{\frac{\pi}{2}} \sqrt{9\cos^2\theta}\cdot 3\cos\theta\,d\theta$$

$$= 4\int_{0}^{\frac{\pi}{2}} 3\cos\theta\cdot 3\cos\theta\,d\theta$$

$$= 36\int_{0}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$\star \quad = 36\int_{0}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 18\int_{0}^{\frac{\pi}{2}} 1+\cos 2\theta\,d\theta$$

$$= 18\left[\theta + \frac{\sin 2\theta}{2}\Big|_{0}^{\frac{\pi}{2}}\right]$$

$$= 18\left[\frac{\pi}{2} + \cancel{\frac{\sin\pi}{2}}^{\,0} - \left(0 + \cancel{\frac{\sin 0}{2}}^{\,0}\right)\right]$$

$$= 18\left[\frac{\pi}{2}\right]$$

$$\boxed{A = 9\pi}$$

NOTE: $9 - \sin^2\theta = (3-\sin\theta)(3+\sin\theta)$

$9\sin^2\theta - 9\cos^2\theta = 9$

$x = 3 = 3\sin\theta$

$1 = \sin\theta$

$\theta = \sin^{-1} 1 = \frac{\pi}{2}$

$x = 0 = 3\sin\theta$

$0 = \sin\theta$

$\sin^{-1} 0 = \theta = 0$

EX2) $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,dx = \int \frac{1}{\sqrt{4-4\sin^2\theta}} \cdot \frac{\frac{2}{3}\cos\theta}{1}\,d\theta = \int \frac{\frac{2}{3}\cos\theta}{\sqrt{4\cos^2\theta}} \cdot d\theta$

$$= \int \frac{\overset{1}{\cancel{2}}}{3}\frac{\cancel{\cos\theta}}{1\,\cancel{2}\,\cancel{\cos\theta}} \cdot d\theta$$

WANT: $9x^2 = 4\sin^2\theta$;

LET: $3x = 2\sin\theta$

$x = \frac{2}{3}\sin\theta \quad \rightarrow \quad \frac{3}{2}x = \sin\theta$

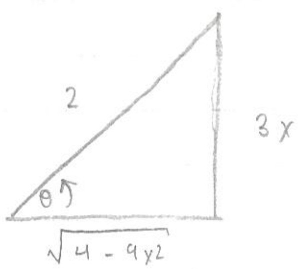
$dx = \frac{2}{3}\cos\theta\,d\theta \qquad \underline{\theta = \sin^{-1}\left(\frac{3x}{2}\right)}$

$$= \frac{1}{3}\int d\theta$$

$$= \frac{1}{3}\left[\,\theta\,\right] + C$$

$$= \frac{\theta}{3} + C$$

$$= \boxed{\frac{1}{3}\sin^{-1}\frac{3x}{2} + C}$$



$\frac{3x}{2} = \sin\theta$

EX 3) $\displaystyle\int \frac{1}{x^2-4x+5}\,dx$

$\displaystyle\int \frac{1}{\underbrace{x^2-4x+4}-4+5}\,dx$

$\displaystyle\int \frac{1}{(x-2)^2+1}\,dx$

WANT $(x-2)^2 = \tan^2\theta$

LET $x-2 = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$$= \int \frac{1}{\tan^2\theta+1}\cdot\frac{\sec^2\theta}{1}\,d\theta = \int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$= \int d\theta$$

$$= \theta + C = \boxed{\tan^{-1}(x-2) + C}$$

NOTE: $12x^2 - 121x + 10$

$12x^2 - 120x - x + 10$

$12x(x-10) - 1(x-10)$

$(12x-1)(x-10)$

a · c		SUM
-120	-1	-121
60	2	62
30	4	34
15	8	23
5	24	29
10	12	22
20	6	[illegible]
60	3	[illegible]

$4x^2 - 8x - 3$

$\underbrace{4x^2 - 8x + 4} - 4 - 3$

$(2x-2)^2 - 7$

EX4) $$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{\sqrt{9\sec^2\theta - 9}}{\cancel{3\sec\theta}} \cdot \cancel{3\sec\theta}\tan\theta\,d\theta = \int \frac{\sqrt{9\tan^2\theta}\,\tan\theta}{1}\,d\theta$$

$$= \int 3\tan^2\theta\,d\theta$$

$$= 3\int \tan^2\theta\,d\theta$$

$$= 3\int (\sec^2\theta - 1)\,d\theta$$

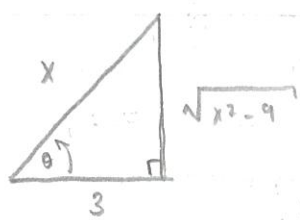
$$= 3\,[\tan\theta - \theta] + C$$

$$= \boxed{3\left[\frac{\sqrt{x^2-9}}{3} - \sec^{-1}\left(\frac{x}{3}\right)\right] + C}$$

WANT: $x^2 = 9\sec^2\theta$

LET: $x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$



$\sec\theta = \frac{x}{3}$

EX5) $$\int e^x\sqrt{1-e^{2x}}\,dx = \int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = \int \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$= \int \cos^2\theta\,d\theta$$

$$= \int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\int 1+\cos 2\theta\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

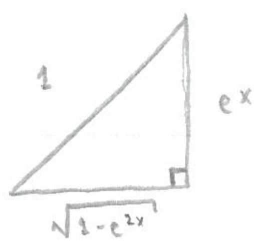
$$= \frac{1}{2}\left[\sin^{-1}(e^x) + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right] + C$$

$$= \boxed{\frac{1}{2}\left[\sin^{-1}(e^x) + e^x\sqrt{1-e^{2x}} + C\right.}$$

WANT: $e^{2x} = \sin^2\theta$

LET: $e^x = \sin\theta \rightarrow \sin^{-1}(e^x) = \theta$

$e^x dx = \cos\theta\,d\theta$



EX6) $$\int_{\sqrt{2}}^{2} \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cancel{\sec\theta}\tan\theta}{\sec^{\cancel{2}}\theta\sqrt{\sec^2\theta-1}}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cancel{\tan\theta}}{\sec\theta\sqrt{\cancel{\tan^2\theta}}}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec\theta}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos\theta\,d\theta$$

$$= \sin\theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \sin\frac{\pi}{3} - \sin\frac{\pi}{4}$$

$$= \boxed{\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}}$$

WANT $x^2 = \sec^2\theta$

LET: $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$x = 2 = \sec\theta$

$\sec^{-1}2 = \theta$

$\theta = \frac{\pi}{3}$

$x = \sqrt{2} = \sec\theta$

$\sec^{-1}(\sqrt{2}) = \theta$

$\theta = \frac{\pi}{4}$

EX7) $$\int \frac{1}{2x^2+4x+7}\,dx$$ NOT FACTORABLE

$$= \int \frac{1}{2x^2+4x+2-2+7}\,dx$$

$$= \int \frac{1}{(\sqrt{2}x+\sqrt{2})^2+5}\,dx$$

WANT: $(\sqrt{2}x+\sqrt{2})^2 = 5\tan^2\theta$

LET: $\sqrt{2}x+\sqrt{2} = \sqrt{5}\tan\theta$

$\sqrt{2}\,dx = \sqrt{5}\sec^2\theta\,d\theta$

$dx = \frac{\sqrt{5}}{\sqrt{2}}\sec^2\theta\,d\theta$

$dx = \frac{\sqrt{10}}{2}\sec^2\theta\,d\theta$

$$= \int \frac{1}{5\tan^2\theta+5}\cdot\frac{\frac{\sqrt{10}}{2}\sec^2\theta}{1}\,d\theta = \frac{\sqrt{10}}{2}\int\frac{\cancelto{1}{\sec^2\theta}}{5\cancel{\sec^2\theta}}\,d\theta = \frac{\sqrt{10}}{10}\int d\theta = \frac{\sqrt{10}}{10}\theta + C$$

$$= \frac{\sqrt{10}}{10}\tan^{-1}\left(\frac{\sqrt{2}x+\sqrt{2}}{\sqrt{5}}\right)+C = \boxed{\frac{\sqrt{10}}{10}\tan^{-1}\left(\frac{\sqrt{10}x+\sqrt{10}}{5}\right)+C}$$