7.5 INTEGRATING RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

06/15

RECALL: $\frac{1}{x}\cdot\frac{x+2}{x+2} - \frac{3}{x+2}\cdot\frac{x}{x}$ LCD $= x(x+2)$

$$= \frac{x+2-3x}{x(x+2)}$$

$$= \boxed{\frac{2-2x}{x(x+2)}}$$

NOTE: IF YOU ARE ASKED TO INTEGRATE

$$\int \frac{(-2x+2)}{x(x+2)}\,dx,$$

YOU WOULD DECOMPOSE THE RATIONAL EXPRESSION

$$\frac{(-2x+2)}{x(x+2)},$$

INTO THE DIFFERENCE

$$\frac{1}{x} - \frac{3}{x+2}.$$

NOW YOU WOULD INTEGRATE

$$\int \left[\frac{1}{x} + \frac{3}{x+2}\right] dx$$

EX 1) INTEGRATE USING PARTIAL FRACTION DECOMPOSITION!

$$\int \left[\frac{-2x+2}{x^2+2x}\right] dx$$

$$\left(\frac{-2x+2}{x^2+2x}\right) = \underbrace{\frac{(-2x+2)}{x(x+2)}}_{\text{THIS IS WHAT WE WILL DECOMPOSE}} = \left[\frac{A}{x} + \frac{B}{(x+2)}\right] \quad \text{LCD: } x(x+2)$$

$$-2x+2 = A(x+2) + Bx$$

$x = 0$; ELIMINATES B

$2 = A(2) + 0$

$2 = 2A$

$A = 1$

$x = -2$; ELIMINATES A

$-2(-2) + 2 = 0 - 2B$

$6 = -2B$

$B = -3$

$$\therefore \int \left[\frac{-2x+2}{x^2+2x}\right] dx$$

$$= \int \frac{1}{x} - \frac{3}{(x+2)}\, dx$$

$$= \ln|x| - 3\ln|x+2| + C$$

$$= \boxed{\ln\left|\frac{x}{(x+2)^3}\right| + C}$$

WE COULD HAVE ALSO SOLVED FOR A & B USING A TECHNIQUE CALLED "EQUATING TERMS."

$$-2x + 2 = A(x+2) + Bx$$

x^1: $-2 = A + B$ USING $A = 1$ $B = -3$

x^0: $2 = 2A \longrightarrow A = 1$

CONSTANTS

HERE ARE SOME EXAMPLES OF SETTING UP THE PARTIAL FRACTION DECOMPOSITIONS.

1) $$\frac{x-2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

- OR -

$$\frac{x-2}{x(x^2+2x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+1}$$

2) $$\frac{x-2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

* 1ST DEGREE, MULT. OF 2

3) $$\frac{2}{(x+1)(x^2+3)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+3)}$$

4) $$\frac{(x+1)}{(x-2)^2(x^2+3)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{(x^2+3)} + \frac{Ex+F}{(x^2+3)^2}$$

5) $$\frac{x+3}{(17x^2-35x+2)} = \boxed{\frac{-\frac{52}{33}}{17x-1} + \frac{\frac{5}{33}}{x-2}}$$

$$(17x-1)(x-2)\left[\frac{x+3}{(17x-1)(x-2)}\right] = \left[\frac{A}{17x-1} + \frac{B}{(x-2)}\right]\cdot(17x-1)(x-2)$$

$$x+3 = A(x-2) + B(17x-1)$$

LET $x = 2$

$5 = 33B$

$B = \frac{5}{33}$

LET $x = \frac{1}{17}$

$\frac{1}{17} + \frac{51}{17} = A\left(\frac{1}{17} - \frac{34}{17}\right)$

$\frac{52}{17} = A\left(-\frac{33}{17}\right)$

$A = -\frac{52}{33}$

NOTE:

$17x^2 - 35x + 2$ $\quad a \cdot c = 34$

$17x^2 - 34x - x + 2$

$17x(x-2) - 1(x-2)$ $\quad -34 \quad -1 \quad$ SUM -35

$(17x-1)(x-2)$

$(x-2)(17x-1)$

$17x(x-2) - 1(x-2)$

$17x^2 - 34x - x + 2$

$17x^2 - 35x + 2$

6) $$\frac{x^2-5}{x^2(x-2)^2} = \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}\right] x^2(x-2)^2$$

$$x^2-5 = Ax\underbrace{(x-2)^2}_{x^2-4x+4} + B\underbrace{(x-2)^2}_{x^2-4x+4} + Cx^2(x-2) + Dx^2$$

LET $x = 0$

$-5 = 4B$

$B = -\frac{5}{4}$

LET $x = 2$

$-1 = 4D$

$D = -\frac{1}{4}$

USING $B = -\frac{5}{4}$

$A = -\frac{5}{4}$

USING $A = \frac{5}{4}$

$C = \frac{5}{4}$

$x^3: \ 0 = A + C$

$x^2: \ 1 = -4A + B - 2C + D$

$x: \ 0 = 4A - 4B \iff 0 = A - B$

$x^0: \ -5 = 4B$

$$\boxed{\frac{x^2-5}{x^2(x-2)^2} = \frac{-\frac{5}{4}}{x} + \frac{-\frac{5}{4}}{x^2} + \frac{\frac{5}{4}}{x-2} + \frac{-\frac{1}{4}}{(x-2)^2}}$$

EX2) $\int \frac{11x+17}{2x^2+7x-4}\,dx$

$= \int \frac{11x+7}{(2x-1)(x+4)}\,dx$

$2x^2+7x-4 \Rightarrow$

a · c
-8
8   -1 → 7
SUM

$2x^2+8x-x-4$

$2x(x+4)-1(x+4)$

$(2x-1)(x+4)$

$$\frac{11x+17}{(2x-1)(x+4)} = \frac{A}{2x-1} + \frac{B}{x+4}$$

$$11x+17 = A(x+4) + B(2x-1)$$

$x=-4 : -27 = -9B \Rightarrow B = 3$

$x = \frac{1}{2} : \frac{11}{2} + \frac{34}{2} = A\left(\frac{1}{2} + \frac{8}{2}\right)$

$\frac{45}{2} = \frac{9}{2}A \Rightarrow A = 5$

- or - $x : 11 = A + 2B$

$11 = A + 6$

$A = 5$

NOTE: CAN COMBINE BOTH METHODS TO FIND ALL VARIABLES

$$= \int \left[\frac{5}{2x-1} + \frac{3}{x+4}\right] dx$$

$$= 5\left(\frac{1}{2}\right)\ln|2x-1| + 3\ln|x+4| + C$$

$$= \boxed{\frac{5}{2}\ln|2x-1| + 3\ln|x+4| + C}$$

EX 3) $\int \frac{x^5 + x^2 + 2}{x^3 - x}\,dx \longrightarrow \int x^2 + 1 + \frac{x^2 + x + 2}{x(x^2-1)}\,dx$

$$= \int x^2 + 1\,dx + \int \frac{x^2 + x + 2}{x(x+1)(x-1)}\,dx$$

$$= \frac{x^3}{3} + x + \int \frac{-2}{x} + \frac{1}{x+1} + \frac{2}{x-1}$$

$$= \boxed{\frac{x^3}{3} + x - 2\ln|x| + \ln|x+1| + 2\ln|x-1| + C}$$

$$x^2 + 1 + \frac{x^2+x+2}{x^3-x}$$

$$x^3 - x \;\big)\; \cancel{x^5} + \cancel{0x^4} + 0x^3 + x^2 + 0x + 2$$

$$-\cancel{x^5} \qquad + x^3$$

$$x^3 + x^2 + 0x + 2$$

$$(-)x^3 \qquad + x$$

$$x^2 + x + 2$$

$$x(x+1)(x-1)\left[\frac{x^2+x+2}{x(x+1)(x-1)}\right] = \left[\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}\right] \cdot x(x+1)(x-1)$$

$$x^2 + x + 2 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

$x = 0 :\ 2 = -A \Rightarrow A = -2$

$x = 1 :\ 4 = 2C \Rightarrow C = 2$

$x = -1 :\ 2 = 2B \Rightarrow B = 1$

EX4) $\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$

$= \int \frac{2x^2 - x + 4}{x(x^2+4)}\,dx = \int \left[\frac{1}{x} + \frac{x-1}{x^2+4} \right] dx = \int \left[\frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4} \right] dx$

let $u = x^2+4$

$du = 2x\,du$

$\frac{1}{2} du = x\,du$

$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$

$2x^2 - x + 4 = A(x^2+4) + Bx^2 + Cx$

$x = 0 : 4 = 4A \Rightarrow A = 1$

$x^0 : 4 = 4A$

$x^1 : -1 = C$

$x^2 : 2 = A + B \Rightarrow \therefore B = 1$

$= \int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{u}\,du - \int \frac{1}{x^2+4}\,dx$

$= \boxed{\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$

EX5) $\int \frac{e^{3x}}{e^{2x}+4}\,dx$

$= \int \frac{(e^x)^3}{(e^x)^2+4}\,dx \longrightarrow \int \frac{u^2}{u^2+4}\,du = \int \frac{(u^2+4)-(4)}{u^2+4}\,du$

** NOTE: $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

let $u = e^x$

$du = e^x\,dx$

$= \int \frac{\cancel{u^2+4}\,1}{\cancel{u^2+4}} - \frac{4}{u^2+4}\,du$

$= \int 1 - \frac{4}{u^2+4}\,du$

$= u - 4 \cdot \frac{1}{2}\tan^{-1}\frac{u}{2} + C$

$= \boxed{e^x - 2\tan^{-1}\frac{e^x}{2} + C}$