7.6 SPECIAL SUBSTITUTIONS

06/19

EX1) $\displaystyle\int \frac{\sqrt{x}}{\sqrt[3]{x}-2}\,dx$

<u>NOTE</u>: $\left.\begin{array}{l}\sqrt{x} = x^{\frac{1}{2}} \\ \sqrt[3]{x} = x^{\frac{1}{3}}\end{array}\right\}$ LCD = 6

$x = u^6$

$dx = 6u^5\,du$

$$= \int \frac{\sqrt{u^6}}{\sqrt[3]{u^6}-2}\cdot\frac{6u^5}{1}\,du$$

$$= \int \frac{u^3}{u^2-2}\cdot\frac{6u^5}{1}\cdot du$$

$$= 6\int \frac{u^8}{u^2-2}\,du$$

$$= 6\int u^6 + 2u^4 + 4u^2 + 8 - \frac{16}{u^2-2}$$

$$= 6\left[\frac{u^7}{7} + \frac{2u^5}{5} + \frac{4u^3}{3} + 8u\right] + 96\int\frac{1}{u^2-2}$$

$$= 6\left[\frac{x^{\frac{7}{6}}}{7} + \frac{2x^{\frac{5}{6}}}{5} + \frac{4x^{\frac{1}{2}}}{3} + 8x^{\frac{1}{6}}\right] + 96\int\frac{\sqrt{2}\sec\theta\tan\theta}{2\sec^2\theta-2}\,d\theta$$

$$u^6 + 2u^4 + 4u^2 + 8 + \frac{16}{u^2-2}$$

$u^2-2\,\overline{)\,u^8 + 0u^7 + 0u^6 + 0u^5 + 0u^4 + 0u^3 + 0u^2 + 0u + 0}$

$(-)\ \underline{u^8 \quad - 2u^6}$

$+2u^6 + 0u^5 + 0u^4$

$(-)\ \underline{2u^6 + 0u^5 - 4u^4}$

$4u^4 + 0u^3 + 0u^2$

$(-)\ \underline{4u^4 + 0u^3 - 8u^2}$

$+8u^2 + 0u + 0$

$(-)\ \underline{+8u^2 + 0u + 16}$

$+16$

$$= \text{" "} + 96\int\frac{\sqrt{2}\sec\theta\cancel{\tan\theta}}{2\tan^{\cancel{2}}\theta}\,d\theta$$

$$= \text{" "} + 48\sqrt{2}\int\frac{\sec\theta}{\tan\theta}\,d\theta$$

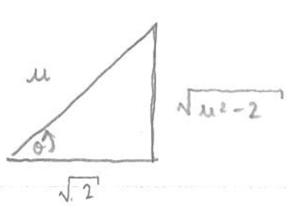
$$= \text{" "} + 48\sqrt{2}\int\csc\theta\,d\theta$$

$$= \text{" "} + 48\sqrt{2}\ln|\csc\theta - \cot\theta| + C$$

WANT : $u^2 = 2\sec^2\theta$

LET: $u = \sqrt{2}\sec\theta$

$du = \sqrt{2}\sec\theta\tan\theta\,d\theta$



$$= \text{" "} + 48\sqrt{2}\ln\left|\frac{u}{\sqrt{u^2-2}} - \frac{\sqrt{2}}{\sqrt{u^2-2}}\right| + C$$

$$= \boxed{\frac{6x^{\frac{7}{6}}}{7} + \frac{12x^{\frac{5}{6}}}{5} + 8x^{\frac{1}{2}} + 48x^{\frac{1}{6}} + 48\sqrt{2}\ln\left|\frac{x^{\frac{1}{6}}-\sqrt{2}}{\sqrt{x^{\frac{1}{3}}-2}}\right| + C}$$

EX2) $\int x\sqrt{x-2}\,dx$

LET: $x-2=u^2 \longrightarrow x=u^2+2$

$dx = 2u\,du$

$= \int (u^2+2)\sqrt{u^2}\;2u\,du$

$= 2\int u^2(u^2+2)\,du$

$= 2\int (u^4+2u^2)\,du = 2\left[\frac{u^5}{5}+\frac{2u^3}{3}\right]+C$

$= \boxed{\frac{2(x-2)^{\frac{5}{2}}}{5}+\frac{4(x-2)^{\frac{3}{2}}}{3}+C}$

EX3) $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$ $\quad x^2\cdot x\,dx$

LET $u^2=x^2+1 \longrightarrow x^2=u^2-1$

$2u\,du = 2x\,dx$

$u\,du = x\,dx$

$= \int \frac{u^2-1}{\cancel{\sqrt{u^2}}}\cdot \cancel{u}^{\,1}\,du$

$= \int u^2-1\,du = \frac{u^3}{3}-u+C$

$= \boxed{\frac{(x^2+1)^{\frac{3}{2}}}{3}-(x^2+1)^{\frac{1}{2}}+C}$

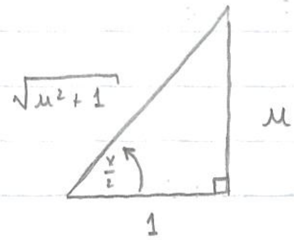
* WEIERSTRAUSS METHOD

EX 4) $\displaystyle\int \frac{1}{\sin x + \cos x}\, dx$

NOTE: CAN BE USED W/ ALL 6 TRIG FUNCTIONS

THE SPECIAL SUBSTITUTION IS

$$\boxed{u = \tan \frac{x}{2}}$$



HOW DOES THIS WORK?

$\sin 2x = 2\sin x \cos x$ $\qquad$ $\cos 2x = \cos^2 x - \sin^2 x$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ $\qquad$ $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$\sin x = 2\left(\frac{u}{\sqrt{u^2+1}}\right)\left(\frac{1}{\sqrt{u^2+1}}\right) = \boxed{\frac{2u}{u^2+1}}$

$\cos x = \left(\frac{1}{\sqrt{u^2+1}}\right)^2 - \left(\frac{u}{\sqrt{u^2+1}}\right)^2 = \boxed{\frac{1-u^2}{u^2+1}}$

WHAT ABOUT dx?

$u = \tan\frac{x}{2}$

$\frac{x}{2} = \tan^{-1} u$

$\frac{1}{2}dx = \frac{1}{u^2+1}du$

$$\boxed{dx = \frac{2}{u^2+1}du}$$

$$= \int \frac{1}{\left(\frac{2u}{\cancel{u^2+1}}\right) + \left(\frac{1-u^2}{\cancel{u^2+1}}\right)} \cdot \frac{2}{\cancel{u^2+1}}\, du$$

$$= \int \frac{2}{2u+1-u^2}\, du$$

$$= -2\int \frac{1}{u^2-2u-1}\, du$$

$$= -2\int \frac{1}{\underbrace{u^2-2u+1}-1-1}\, du$$

$$= -2\int \frac{1}{(u-1)^2-2}\, du$$

WANT: $(u-1)^2 = 2\sec^2\theta$

LET: $u - 1 = \sqrt{2}\sec\theta$

$du = \sqrt{2}\sec\theta\tan\theta\, d\theta$

$$= -2\int \frac{\sqrt{2}\sec\theta\tan\theta}{2\sec^2\theta - 2}\, d\theta$$

$$= -\cancel{2}\int \frac{\sqrt{2}\sec\theta\cancel{\tan\theta}}{\cancel{2}\tan^{\cancel{2}}\theta}\, d\theta$$

$$= -\sqrt{2}\int \csc\theta\, d\theta$$

$$= -\sqrt{2}\ln|\csc\theta - \cot\theta| + C$$

$$= -\sqrt{2}\ln\left|\frac{u-1-\sqrt{2}}{\sqrt{u^2-2u-1}}\right| + C$$

$$= \boxed{-\sqrt{2}\ln\left|\frac{\tan\frac{x}{2}-1-\sqrt{2}}{\sqrt{\tan^2\frac{x}{2}-2\tan\frac{x}{2}-1}}\right| + C}$$

HWK #69) $\int \frac{1}{\tan x + \sin x}$

$= \int \frac{1}{\frac{2u}{1-u^2} + \frac{2u}{1+u^2}} \cdot \frac{2}{u^2+1}\, du$

$= \int \frac{2}{\frac{2u(u^2+1)}{1-u^2} + 2u}\, du$

$= \int \left[\frac{2}{\frac{2u(u^2+1)}{1-u^2} + 2u}\right] \cdot \left[\frac{1-u^2}{1-u^2}\right] du$

$= \int \frac{\cancel{2}(1-u^2)}{\cancel{2}u(u^2+1) + \cancel{2}u(1-u^2)}\, du$

$= \int \frac{1-u^2}{u(u^2+1) + u(1-u^2)}\, du$

$= \int \frac{1-u^2}{\cancel{u^3} + u + u - \cancel{u^3}}\, du$

$= \int \frac{1-u^2}{2u}\, du$

$= \int \left[\frac{1}{2u} - \frac{u}{2}\right] du = \frac{1}{2} \ln |u| - \frac{u^2}{4} + C$

$= \boxed{\frac{1}{2} \ln \left|\tan \frac{x}{2}\right| - \frac{\tan^2 \frac{x}{2}}{4} + C}$