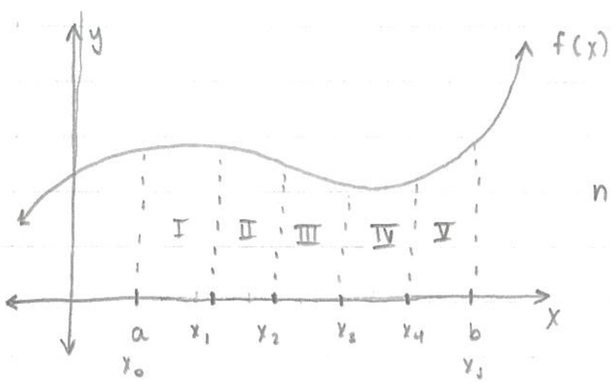
7.7 NUMERICAL INTEGRATION; SIMPSON'S RULE

06/19

<u>NOTE</u>: MANY FUNCTIONS CANNOT BE INTEGRATED, THEREFORE WE USE A NUMERICAL APPROXIMATION TO ESTIMATE THE AREA UNDER THE CURVE.

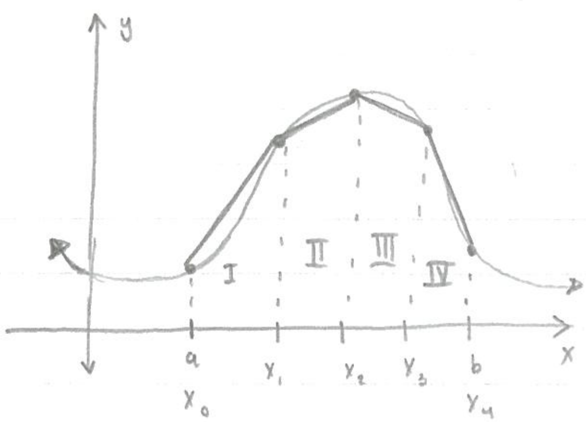
<u>RECALL</u>: $$\int_a^b f(x)\,dx = \lim_{n \to +\infty} \sum_{k=1}^{n} \underbrace{f(x_k^*)}_{\text{HEIGHT}} \underbrace{\Delta x}_{\text{WIDTH}}$$

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$



$n = 5$

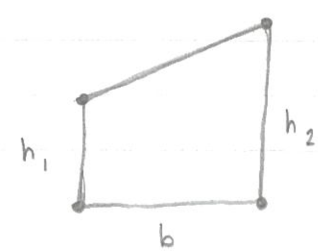
SUPPOSE WE USED TRAPAZOIDS TO ESTIMATE AREA



$$\Delta x = \frac{b-a}{n}$$

$$n = 4$$

$$A_{\text{TRAPEZOID}} = \left[\frac{h_1 + h_2}{2}\right] \cdot b$$



$$\int_a^b f(x)\,dx \approx \left[\frac{f(x_0)+f(x_1)}{2}\right]\Delta x + \left[\frac{f(x_1)+f(x_2)}{2}\right]\Delta x + \left[\frac{f(x_2)+f(x_3)}{2}\right]\Delta x + \left[\frac{f(x_3)+f(x_4)}{2}\right]$$

$$\approx \frac{\Delta x}{2}\left[f(x_0) + 2(f x_1) + 2(f x_2) + 2f(x_3) + 2(f x_4)\right]$$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2(f x_1) + 2(f x_2) + 2f(x_3) + 2f(x_4)\right]$$

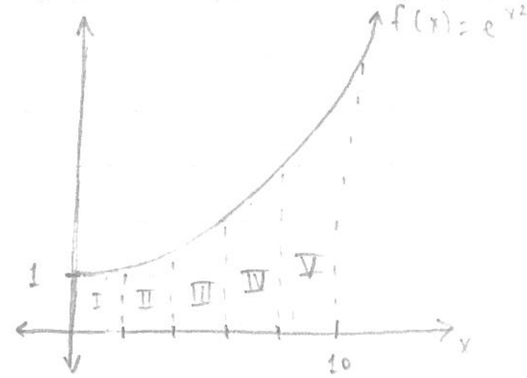
TRAPAZOIDAL RULE:

$$\star \quad \boxed{\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2(f x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]}$$

EX1) USE THE TRAPAZOIDAL RULE W/ $n = 5$ TO APPROXIMATE:

$$\int_0^{10} e^{x^2}\,dx =$$

$$\Delta x = \frac{b-a}{n} = \frac{10-0}{5} = 2$$

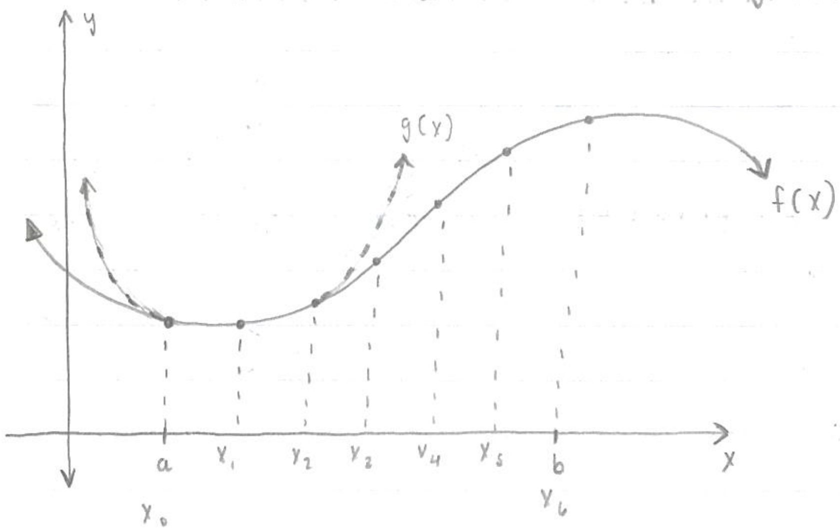


$$\int_0^{10} e^{x^2}\,dx = \frac{10-0}{2(5)}\left[f(0) + 2f(2) + 2f(4) + 2f(6) + 2f(8) + f(10)\right]$$

$$= e^0 + 2e^4 + 2e^{16} + 2e^{36} + 2e^{64} + e^{100}$$

$$= \boxed{2.6881 * 10^{43}}$$

SIMPSON'S RULE: (very good at capturing concavity by using parabolas!)



$$g(x_0) = f(x_0) \qquad g(x_0) = ax_0 + bx_0 + c$$
$$g(x_1) = f(x_1) \qquad g(x_1) = ax_1 + bx_1 + c$$
$$g(x_2) = f(x_2) \qquad g(x_2) = ax_2 + bx_2 + c$$

SINCE $g(x) = ax^2 + bx + c$, WE WILL APPROXIMATE

$$\int_{x_0}^{x_2} f(x)\,dx \quad \text{WITH} \quad \int_{x_0}^{x_2} g(x)\,dx \qquad \underline{n = 2}$$

$$\int_{x_0}^{x_2} g(x)\,dx = \int_{x_0}^{x_2} (Ax^2 + Bx + C)\,dx$$

$$= \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \Big|_{x_0}^{x_2}$$

$$= \frac{A}{3}(x_2^3 - x_0^3) + \frac{B}{2}(x_2^2 - x_0^2) + C(x_2 - x_0)$$

$$= (x_2 - x_0)\left[\frac{A}{3}(x_2^2 + x_2x_0 + x_0^2) + \frac{B}{2}(x_2 + x_0) + C\right]$$

$$= \underbrace{\frac{x_2 - x_0}{6}}_{\frac{x_2 - x_0}{3n}} \left[\underbrace{2A(x_2^2 + x_2x_0 + x_0^2) + 3B(x_2 + x_0) + 6C}_{g(x_0) + 4g(x_1) + g(x_2)}\right]$$

$\underline{\text{NOTE}}$: $x_1 = \dfrac{x_2 + x_0}{2}$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$f(x_0) + 4f(x_1) + f(x_2)$$

SIMPSON'S RULE:

$$\boxed{\int_a^b \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4(fx_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]}$$