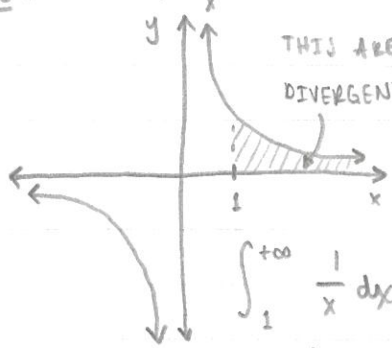
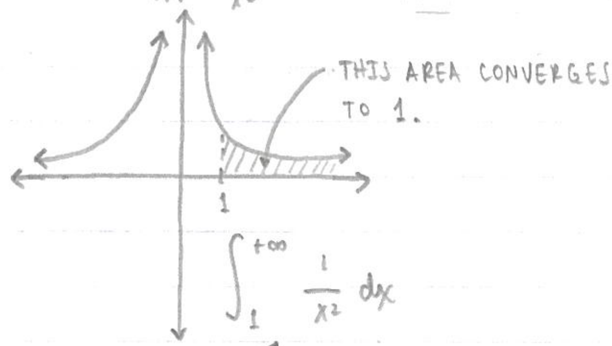
7.8 IMPROPER INTEGRALS

RECALL: $f(x) = \frac{1}{x}$ $\qquad$ $f(x) = \frac{1}{x^2}$





$$\int_1^{+\infty} \frac{1}{x}\,dx \qquad \int_1^{+\infty} \frac{1}{x^2}\,dx$$

IMPROPER INTEGRAL

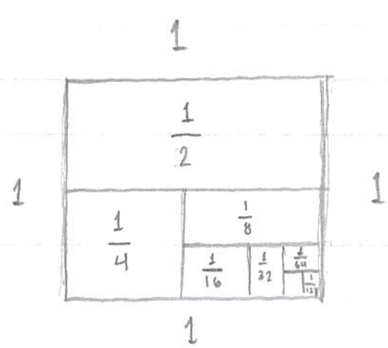
EX 1) $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$

$$= \lim_{a\to+\infty} \int_1^{a} \frac{1}{x^2}\,dx$$

$$= \lim_{a\to+\infty} \left[-\frac{1}{x} \Big|_1^a \right] = \lim_{a\to+\infty} \left[-\frac{1}{a} - (-1) \right]$$

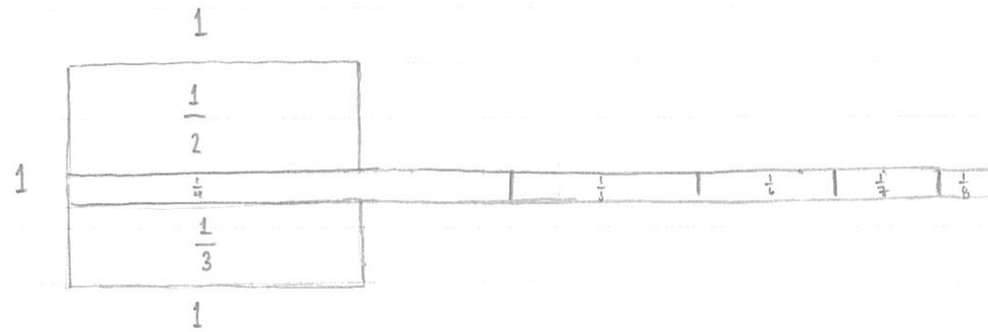
$$= \lim_{a\to+\infty} \left[-\cancelto{0}{\frac{1}{a}} + 1 \right] = 1 \text{ CONVERGENT}$$

HERE'S AN EXAMPLE OF AN INFINITE SUM CONVERGING TO AN AREA OF 1



$$\underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots}_{\text{THE SUM CONVERGES TO 1}}$$

HERE'S AN EXAMPLE OF A DIVERGENT SUM:



$$\underbrace{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots}$$

THIS SUM IS DIVERGENT. IT IS CONTINUOUSLY GROWING; GETTING CLOSER TO INFINITY!

EX 2) $$\int_1^{+\infty}\frac{1}{x}\,dx = \lim_{a\to+\infty}\int_1^a \frac{1}{x}\,dx$$

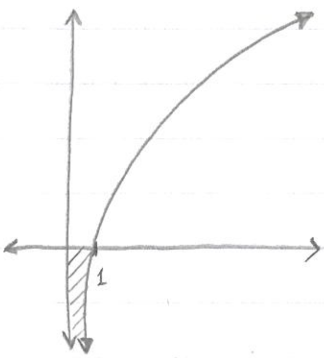
$$= \lim_{a\to+\infty} \ln x \Big|_1^a$$

$$= \lim_{a\to+\infty} \ln a - \underbrace{\ln 1}_{0}$$

$$= \lim_{a\to+\infty} \ln a$$

$$= +\infty \text{ DIVERGENT}$$

EX 3) $\int_0^1 \ln x \, dx \longrightarrow = \lim_{a \to 0^+} \int_a^1 \ln x \, dx$



$$= \lim_{a \to 0^+} \left[x \ln x - x \Big|_a^1 \right]$$

$$= (1 \ln 1 - 1) - \lim_{a \to 0^+} \left[a \ln a - a \right]$$

$$= -1 - \lim_{a \to 0^+} a \ln a + \cancel{\lim_{a \to 0^+} a}^{\,0}$$

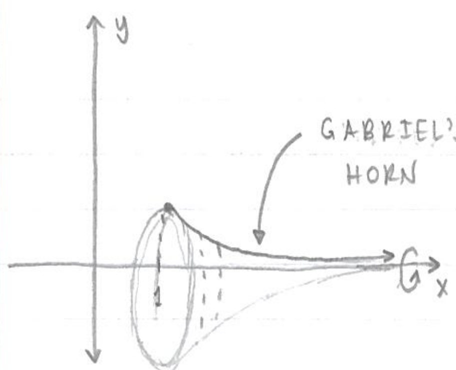
$$= -1 - \lim_{a \to 0^+} \frac{\ln a}{\frac{1}{a}}$$

$$\overset{L'H}{=} -1 - \lim_{a \to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}}$$

$$= -1 - \cancel{\lim_{a \to 0^+} (-a)}^{\,0}$$

$$= -1 \quad \text{CONVERGENT}$$

EX4) REVOLVE THE AREA UNDER $f(x) = \frac{1}{x}$ ON $[1, +\infty)$ ABOUT THE X-AXIS.



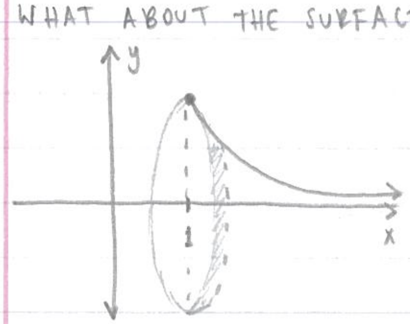
$$V = \pi \int_1^{+\infty} \left[\frac{1}{x} \right]^2 dx$$

$$= \pi \underbrace{\lim_{a \to +\infty} \int_1^a \frac{1}{x^2} \, dx}$$

$$= \pi \cdot 1$$

$$= \pi \quad \text{CONVERGENT}$$

WHAT ABOUT THE SURFACE AREA OF GABRIEL'S HORN?



★ $SA = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} \, dx$

$$= 2\pi \int_1^{+\infty} \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\,dx$$

$$= 2\pi \lim_{a\to+\infty} \int_1^{a} \frac{\sqrt{1 + \frac{1}{x^4}}}{x}\,dx$$

$$= +\infty$$ DIVERGENT

EX5) $$\int_{-\infty}^{0} xe^x\,dx = \lim_{a\to-\infty}\int_a^0 xe^x dx = \lim_{a\to-\infty}\left[xe^x - e^x \right|_a^0$$

$$= \left[0 - e^0\right] - \lim_{a\to-\infty}\left[ae^a - e^a\right]$$

	u	dv
(+)	x	e^x
(-)	1	e^x
(+)	0	e^x

←

$$= (-1) - \lim_{a\to-\infty}\left[\frac{a}{e^{-a}}\right] + \lim_{a\to-\infty} e^a \xrightarrow{} 0$$

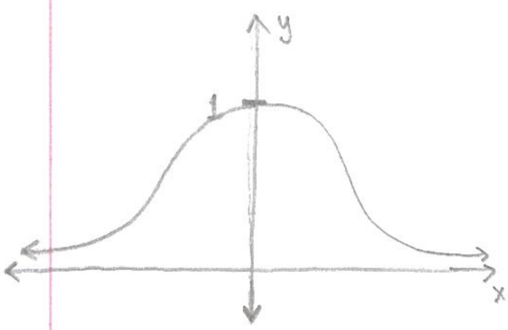
L'H
$$= -1 - \lim_{a\to-\infty}\left[\frac{1}{-e^{-a}}\right]$$

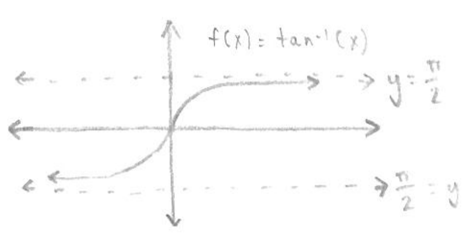
$$= -1 - \lim_{a\to-\infty}\left[-e^a\right] \xrightarrow{} 0$$

NOTE: INFINITE MEANS NOT FINITE

$$= -1$$ CONVERGENT

EX6) $$\int_{-\infty}^{+\infty}\frac{1}{x^2+1}\,dx = \int_{-\infty}^{0}\frac{1}{x^2+1}\,dx + \int_0^{+\infty}\frac{1}{x^2+1}\,dx = \lim_{a\to-\infty}\int_a^0\frac{1}{x^2+1}\,dx + \lim_{b\to+\infty}\int_0^b\frac{1}{x^2+1}\,dx$$





$$= \lim_{a\to-\infty}\tan^{-1}x\Big|_a^0 + \lim_{b\to+\infty}\tan^{-1}x\Big|_0^b$$

$$= \cancel{\tan^{-1}(0)}^{\,0} - \lim_{a\to-\infty}\tan^{-1}(a) + \lim_{b\to+\infty}\tan^{-1}(b) - 0$$

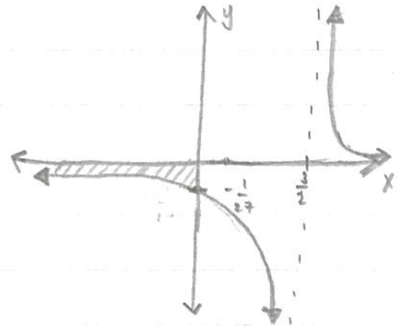
$$= -\lim_{a\to-\infty}\tan^{-1}(a) + \lim_{b\to+\infty}\tan^{-1}(b)$$

$$= -\left[-\frac{\pi}{2}\right] + \left[\frac{\pi}{2}\right]$$

$$= \pi$$ CONVERGENT

*NOTE: IF VALUE IN RANGE OF INTEGRATION IS NOT IN FUNCTION DOMAIN, BREAK UP THE INTEGRAL

EX7) $\int_{-\infty}^{0} \frac{1}{(2x-3)^3}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{1}{(2x-3)^3}\,dx = \frac{1}{2}\lim_{a\to-\infty}\int_{2a-3}^{-3}\frac{1}{u^3}\cdot du$



LET $u = 2x-3$

$du = 2x\,dx$

$dx = \frac{1}{2}\,du$

$x = 0 \to u = -3$

$x = a \to u = 2a-3$

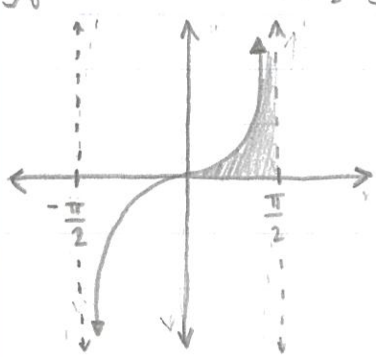
$$= \frac{1}{2}\lim_{a\to-\infty}\left[-\frac{1}{2u^2}\Big|_{2a-3}^{-3}\right]$$

$$= -\frac{1}{2}\lim_{a\to-\infty}\left[\frac{1}{2(-3)^2} - \frac{1}{2(2a-3)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{18}\right] + \cancelto{0}{\frac{1}{4}\lim_{a\to-\infty}\left[\frac{1}{(2a-3)^3}\right]}$$

$$= \boxed{-\frac{1}{36}}$$ CONVERGENT

EX8) $\int_0^{\frac{\pi}{2}} \tan x\,dx = \lim_{a\to\frac{\pi}{2}^-}\int_0^a \tan x\,dx = \lim_{a\to\frac{\pi}{2}^-}\left[-\ln(\cos x)\Big|_0^a\right]$



$$= -\lim_{a\to\frac{\pi}{2}^-}\left[\ln(\cos a)\right] - \cancelto{0}{\ln(\cos 0)}$$

$$= -\ln\left\{\lim_{a\to\frac{\pi}{2}^-}\cancelto{0}{(\cos a)}\right\}$$

$$= -\ln 0 = +\infty$$ DIVERGENT

$\downarrow$

$-1\cdot-\infty$

EX9) $\int_1^{+\infty}\frac{\ln x}{x^2}\,dx = \lim_{a\to+\infty}\int_1^a \frac{\ln x}{x^2}\,dx$

	u	dv
(+)	$\ln x$	$\frac{1}{x^2}$
(−)	$\frac{1}{x}$	$-\frac{1}{x}$

$$= \lim_{a\to+\infty}\left[-\frac{\ln x}{x}\Big|_1^a\right] + \int_1^a \frac{1}{x^2}\,dx$$

$$= \lim_{a\to+\infty}\left[\frac{-\ln a}{a} + \cancelto{0}{\frac{\ln 1}{1}}\right] + \lim_{a\to+\infty}\left[-\frac{1}{x}\Big|_1^a\right]$$

$$= -\lim_{a\to+\infty}\left[\frac{\ln a}{a}\right] + \lim_{a\to+\infty}\left[-\cancelto{0}{\frac{1}{a}} + \frac{1}{1}\right]$$

$$\overset{L'H}{=} -\cancelto{0}{\lim_{a\to+\infty}\left[\frac{\frac{1}{a}}{1}\right]} + 1$$

$$= 1$$ CONVERGENT