§7-1 COUNTING

Definition

> **<u>The Multiplication Principle</u>**
> If one experiment can result in *m* possible outcomes and another experiment can result in *n* possible outcomes then there are *mn* possible outcomes of the two experiments.

Example 1 If you have four different shirts and three different pairs of pants, how many different outfits can you put together?

Solution There are four possible choices of shirts and three possible choices for pants so by the multiplication principle there are $4 \times 3 = 12$ possible outfits.

Example 2 If you have three types of bread, two types of cheese and four kinds of meat then, how many different types of sandwich can you make?

Solution There are $3 \times 2 \times 4 = 24$ possible kinds of sandwich.

Example 3 There are three roads connecting Apple City to Banana Junction, and 4 roads connecting Banana Junction to Cantaloupe Village. If these are the only roads available, how many different ways are there to get from Apple City to Cantaloupe Village?

Solution There are $3 \times 4 = 12$ possible paths.

Definition

> $n! = n \cdot (n-1) \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$ is called *n* **factorial**.

Example 4 Evaluate 6!

Solution $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Example 5 Find the total number of ordered arrangements of the three different letters *x*, *y*, and *z*.

Solution You may want to ask yourself, how many ordered arrangements begin with *x*? How many begin with *y* and *z*? You should obtain the following six different ordered arrangements.

xyz	*yxz*	*zxy*
xzy	*yzx*	*zyx*

There are no other possible ordered arrangements other than those listed above.

Note that for the first position we could have chosen any of the three letters. For the second position we have only two letters to choose from and the single remaining letter is placed in the third position. Therefore there are $3! = 3 \times 2 \times 1 = 6$ possible ordered arrangements or *permutations*.

Definition

> A **permutation** is an <u>ordered</u> arrangement of *n* objects.

Examples using three different types of permutation follow.

Definition

> **<u>Type I Permutation</u>**
> The number of permutations that can made from n *distinguishable objects* is equal to $n!$

Example 6 How many different batting orders are possible for a little league baseball team that consists of 9 players?

Solution There are $9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362,880$ different batting orders that can be formed.

Example 7 How many different ways can you arrange the letters in the word BRIDGE? (Note that all the letters in the word BRIDGE are different.)

Solution There are $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ different letter arrangements.

Next we wish to determine how many different ordered arrangements of r objects can be formed from n objects.

Definition

> **<u>Type II Permutation</u>**
> The number of ordered arrangements of r objects taken from n objects (called a permutation of r objects taken n at a time) is equal to ${}_nP_r = \dfrac{n!}{(n-r)!}$.

Example 8 Suppose you have 13 different colored Easter eggs and there is only enough room to display five of them on your shelf. How many different ways can this be done?

Solution There are ${}_{13}P_5 = \dfrac{13!}{(13-5)!} = \dfrac{13!}{8!} = 154,440$ different ways you can display five of the Easter eggs on your shelf.

Example 9 How many ways can you select a president, vice-president and a secretary from a group of 11 people?

Solution There are ${}_{11}P_3 = \dfrac{11!}{(11-3)!} = \dfrac{11!}{8!} = 990$ different ways you can select a president, vice-president and a secretary.

It should be noted that the two types of permutations above deal with n distinguishable objects. The next type of permutation deals with the case in which some of the objects are alike.

Definition

> **<u>Type III Permutation</u>**
> The number of ordered arrangements (permutations) in which $r_1, r_2, r_3 ...$ are alike is equal to $\dfrac{n!}{(r_1!)(r_2!)(r_3!)...}$.

Example 10 How many different ordered arrangements or permutations can be made using all the letters in the word Mississippi?

Solution There is one m, 4 *i*'s, 4 *s*'s and 2 *p*'s. Therefore there are $\dfrac{11!}{(1!)(4!)(4!)(2!)} = 34,650$ different permutations that can be made from Mississippi.

Example 11 How many different permutations of the letters in the word statistics are there?

Solution There are $\frac{10!}{(3!)(3!)(1!)(2!)(1!)} = 50,400$ different permutations that can be made.

Suppose we wish to determine how many different groups of r objects can be formed from n objects where we are not concerned with any order arrangement within the group of r objects. These types of problems, where order does not matter, are called combination problems.

Definition

> A **combination** is a non-ordered arrangement of n objects taken r at a time. The number of different groups of size r that can be formed from n objects is calculated by the formula ${}_nC_r = \frac{n!}{r!(n-r)!}$.

Example 12 How many committees of 4 people can be formed from a group of 20 people?

Solution ${}_{20}C_4 = \frac{20!}{(4!)(20-4)!} = \frac{20!}{(4!)(16!)} = 4845$ possible committees

Example 13 How many different groups of three boys and two girls can be formed from a class of nine boys and seven girls.

Solution First we calculate the number of groups of three boys chosen out of nine:

$$ {}_9C_3 = \frac{9!}{(3!)(9-3)!} = \frac{9!}{(3!)(6!)} = 84 \text{ possible groups of three boys} $$

Now we calculate the number of groups of two girls chosen out of seven.

$$ {}_7C_2 = \frac{7!}{(2!)(7-2)!} = \frac{7!}{(2!)(5!)} = 21 \text{ possible groups of two girls} $$

The *multiplication principle* tells us that the number of possible combinations of these two groups is $84 \times 21 = 1764$. So there are 1764 possible groups of three boys.

Example 14 How many poker hands consisting of two aces and three kings are possible?

Solution First we calculate the number of ways to choose two aces out of the four in a poker deck.

$$ {}_4C_2 = \frac{4!}{(2!)(4-2)!} = \frac{4!}{(2!)(2!)} = 6 \text{ ways to pick two aces} $$

Now we calculate the number of ways to choose three aces out of the four in a deck.

$$ {}_4C_3 = \frac{4!}{(3!)(4-3)!} = \frac{4!}{(3!)(1!)} = 8 \text{ ways to pick three kings} $$

The *multiplication principle* tells us that the number of possible combinations of these two groups is $6 \times 8 = 48$. So there are 48 possible poker hands consisting of two aces and three kings.

§7-1 PROBLEM SET

Solve each problem.

1. A deli offers a choice of three types of bread, four kinds of cheese and four different meats for sandwiches. How many different sandwiches is it possible to order?

2. In a three digit number, the hundreds digit can be 2, 4, 6 or 8, the tens digit can be 3, 5 or 7, and the ones digit can be 1, 3, or 9. How many numbers fit this description?

3. John has 5 shirts and 4 pairs of pants. How many different outfits can he make from this selection?

4. There are four roads from Pointville to Lineton and three roads from Lineton to Planeville. How many routes are there from Pointville to Planeville passing through Lineton?

Evaluate each factorial expression.

5. $3!$ **6**. $4!$ **7**. $5!$ **8**. $8!$

9. $\frac{8!}{7!}$ **10**. $\frac{12!}{11!}$ **11**. $\frac{10,000!}{9998!}$ **12**. $\frac{10!}{(4!)(6!)}$

Evaluate where ${}_nP_r = \frac{n!}{(n-r)!}$ and ${}_nC_r = \frac{n!}{r!(n-r)!}$.

13. ${}_5P_2$ **14**. ${}_5P_5$ **15**. ${}_5C_2$ **16**. ${}_5C_5$

17. ${}_{14}P_{13}$ **18**. ${}_{5000}P_1$ **19**. ${}_{14}C_{13}$ **20**. ${}_{5000}C_1$

How many different ways can the letters of these words be rearranged?

21. CAT **22**. DUCK **23**. MOUSE

24. MOOSE **25**. CHEESES **26**. BOOKKEEPER

Solve each problem.

27. How many ways can four people be chosen from a group of fifteen?

28. How many ways can five cards be chosen from a deck of fifty-two?

29. How many ways can a president, vice-president, and treasurer be elected from a class of thirty students?

30. Question for thought: Is the combination for a combination lock a permutation or a combination? Why?

§7-1 PROBLEM SOLUTIONS

1.	48	**2**.	36	**3**.	20	**4**.	12
5.	6	**6**.	24	**7**.	120	**8**.	40,320
9.	8	**10**.	12	**11**.	99,990,000	**12**.	210
13.	20	**14**.	120	**15**.	10	**16**.	1
17.	87,178,291,200			**18**.	5000	**19**.	14
20.	5000	**21**.	6	**22**.	24	**23**.	120
24.	60	**25**.	420	**26**.	151,200	**27**.	1365
28.	2,598,960	**29**.	24,360				

30. The numbers of a “combination” lock must be entered in a specific order so it might be more appropriate to call it a “permutation” lock.